

How to Calculate the Square Root of Any Number by Hand

Calculating square roots by hand is not even what a practicing mathematician would call a leisurely pastime. The procedure is hardly, if ever, taught in school in this age of computers. It is, however, quite possible to determine the square root of a number to any degree of accuracy for which a person is willing to stretch the work, and the mathematics involved in the procedure is not complex. A brief glimpse of a hand-calculated square root problem appears very much like a long division problem, and some aspects of the procedure are similar.

Step One – The set-up

To illustrate these steps we will find the square root of 1,234.56

- a) Write the number for which the square root is to be calculated under a square root sign ($\sqrt{\quad}$), elongated if desired to simulate the appearance of long division. As you are writing the number under the radical sign, mark off two-digit pairs on both sides of the decimal point in the original number. Use a 'hash' mark (looks just like an apostrophe) to pair off adjacent digits of the number. It is important that you start this "pairing off" from the decimal point. On the whole number side (the left side of the decimal point), cease pairing off digits when there are no digits or one digit left at the front of the number. On the fractional side (the right side of the decimal point), pair off digits using zeros as placeholders if necessary.

$$\sqrt{12'34.56'00'00'00}$$

- b) Locate and mark the decimal point of this number. Above the radical sign, place another decimal point directly above the decimal point of the original number. This second point will become the decimal point in the square root of the original number. Note that since the handwritten work will look very similar to a long division problem, care should be given in allowing for space to write products, take differences, and carry down digits just as in long division.

$$\sqrt{12'34.56'00'00'00}$$

- c) Each pair of digits in the original number will represent ONE digit in the answer – in others words, it takes two digits in the original to make one digit of your answer. There is one exception – when there is an odd number of digits on the whole number side of the original number. Each pairing of digits on the fractional (right) side will constitute one digit of accuracy for the final square root calculation, so if exceptional accuracy is desired, leave plenty of room on the right side of the paper for additional work. In the above example as it is presented, one would be able to calculate the square root of 1234.56 to four decimal places beyond the decimal point. As many additional pairs of zeroes could be added to the end of the number as one wished to calculate.

Step Two – The First Calculation

- a) Look at the very first pairing at the front of the number, similar in a sense to the way one looks at a number when beginning long division. This first pairing will either consist of one or two digits — for procedural purposes, it makes no difference.

In our example, the first pairing is '12'. Determine the largest perfect square number that can be subtracted from this leading pair.

$$\sqrt{12' 34 . 56' 00' 00' 00}$$

PERFECT SQUARES
1 x 1 = 1
2 x 2 = 4
3 x 3 = 9
4 x 4 = 16
5 x 5 = 25
6 x 6 = 36
7 x 7 = 49
8 x 8 = 64
9 x 9 = 81

In this case, the largest perfect square less than 12 is 9

PERFECT SQUARES
1 x 1 = 1
2 x 2 = 4
3 x 3 = 9
4 x 4 = 16
5 x 5 = 25
6 x 6 = 36
7 x 7 = 49
8 x 8 = 64
9 x 9 = 81

- b) Write this squared number underneath the first pairing in preparation for subtraction, and write the squared root of this perfect square above the radical sign, **directly** above the first pairing.

$$\begin{array}{r} 3 \\ \sqrt{12' 34 . 56' 00' 00' 00} \\ \underline{9} \end{array}$$

- c) Subtract the perfect square from the value of the first pairing and write the difference below (12 - 9 = 3)

$$\begin{array}{r} 3 \quad \bullet \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ 3 \end{array}$$

- d) Bring down the next pair of digits in the original number (34) next to the difference that was just found (3), thus forming the number: **334**

$$\begin{array}{r} 3 \quad \bullet \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ 3 \quad 34 \end{array}$$

- e) Draw a vertical line to the left of the number: **334**

$$\begin{array}{r} 3 \quad \bullet \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ \boxed{3 \quad 34} \end{array}$$

So far, the look of the problem should seem very much like a long division problem as promised. It helps to conform to this style of presentation, since keeping track of differences and products will be as vital here as in division.

Step Three – Determining the Trial and Actual Divisors

The mathematics involved thus far has been relatively simple. The first spot of good news is that it only gets *slightly* more difficult in this third step. The second spot of good news is that with this somewhat more lengthy step, the procedure will begin to repeat indefinitely, until the degree of accuracy desired is finally obtained or the answer comes out even (with no remainder).

- a) Whenever you draw a vertical line next to a partial dividend (NOTE: this was the last thing you did in step 2), multiply whatever number is above the radical sign by **20**. In this case, that means multiplying **3** times **20** which equal **60**. This number is known as a **trial divisor**.
- b) Place the trial divisor to the left of the vertical line. Understand that because you are always going to multiply by 20, you will always have a zero at the right-hand end of the trial divisor. This zero will not stay zero in the actual divisor.

$$\begin{array}{r} 3 \quad \bullet \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ \boxed{3 \quad 34} \end{array}$$

trial divisor → **60** | 3 34

- c) Its now time to convert the trial divisor into the actual divisor. Estimate how many times the first digit in the trial divisor will go into the first two digits of the partial dividend (334).

In this case, 6 will divide into 33, 5 times. Hence, the actual divisor will divide into 334 either 5, or one less, 4 times.

TO MAKE THE ACTUAL DIVISOR, YOU PLACE THE ESTIMATED NUMBER OF TIMES THE TRIAL DIVISOR WILL DIVIDE (in this case, 5) ABOVE THE RADICAL SIGN DIRECTLY CENTERED OVER THE NEXT PAIR OF DIGITS (in this case, 34) AND ADD 5 TO THE TRIAL DIVISOR. THUS, THE ACTUAL DIVISOR BECOMES 65.

$$\begin{array}{r}
 3 \quad 5 \bullet \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \left| \begin{array}{l} 3 \quad 34 \\ \hline 3 \quad 25 \end{array} \right.
 \end{array}$$

- d) The newly formed actual divisor **MUST** be tested. In this case, this means multiplying 5 times 65.

$$\begin{array}{r}
 65 \\
 \times 5 \\
 \hline
 325
 \end{array}$$

In our given problem 325 is less than 334, so we are done finding this actual divisor. But if 5 times 65 had produced a number **LARGER** than 334, we would (a) place 4 above the radical sign and (b) change the actual divisor into 64. Then we would multiply 4 times 64.

In other words, the trial divisor changes into the actual divisor by **ADDING** the estimated number of times that it will divide into the partial dividend.

- e) Once the actual divisor is determined, place the product of the actual divisor and estimated number ($5 \times 65 = 325$) underneath the partial dividend and subtract.

$$\begin{array}{r}
 3 \quad 5 \bullet \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \left| \begin{array}{l} 3 \quad 34 \\ \hline 3 \quad 25 \\ \hline 9 \end{array} \right.
 \end{array}$$

Step Four – Finishing Off the Answer

Here is where the steps begin to repeat.

- a) **Bring down** the next pair of digits and **place** them next to the difference obtained in the above step.

$$\begin{array}{r}
 3 \quad 5 \bullet \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \left| \begin{array}{r} 3 \quad 34 \\ 3 \quad 25 \\ \hline 9 \quad 56 \end{array} \right.
 \end{array}$$

- b) **Draw a vertical line** to the left of the number: **956**

$$\begin{array}{r}
 3 \quad 5 \bullet \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \left| \begin{array}{r} 3 \quad 34 \\ 3 \quad 25 \\ \hline 9 \quad 56 \end{array} \right.
 \end{array}$$

- c) Whenever you draw a vertical line next a partial dividend, multiply **whatever** number is above the radical sign by **20**. In this case that means multiplying **35** times **20** which equal **700**. This number is known as a **trial divisor**. NOTE: Every time you bring down a pair of digits and go up above the radical sign and multiply by 20, the number above the sign will be larger than previous time you did this.

- d) Place the trial divisor to the left of the vertical line.

$$\begin{array}{r}
 3 \quad 5 \bullet \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \left| \begin{array}{r} 3 \quad 34 \\ 3 \quad 25 \\ \hline 9 \quad 56 \end{array} \right. \\
 700 \left| \begin{array}{r} 9 \quad 56 \end{array} \right.
 \end{array}$$

- e) Its now time to convert the trial divisor into the actual divisor. Estimate how many times the first digit in the trial divisor will go into the first digit of the partial dividend (956).

In this case, 7 will divide into 9, 1 times.

TO MAKE THE ACTUAL DIVISOR, YOU PLACE THE ESTIMATED NUMBER OF TIMES THE TRIAL DIVISOR WILL DIVIDE (in this case, 1) ABOVE THE RADICAL SIGN DIRECTLY CENTERED OVER THE NEXT PAIR OF DIGITS (in this case, 56) AND ADD 1 TO THE TRIAL DIVISOR. THUS, THE ACTUAL DIVISOR BECOMES 701.

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \quad \left| \begin{array}{l} 3 \quad 34 \\ 3 \quad 25 \end{array} \right. \\
 701 \quad \left| \begin{array}{l} 9 \quad 56 \end{array} \right.
 \end{array}$$

- f) The newly formed actual divisor **MUST** be tested. In this case, this means multiplying 1 times 701

$$\begin{array}{r}
 701 \\
 \times 1 \\
 \hline
 701
 \end{array}$$

In our given problem 701 is less than 956, so we are done finding this actual divisor.

- g) Once the actual divisor is determined, place the product of the actual divisor and estimated number (1 x 701 = 701) underneath the partial dividend and subtract.

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \quad \left| \begin{array}{l} 3 \quad 34 \\ 3 \quad 25 \end{array} \right. \\
 701 \quad \left| \begin{array}{l} 9 \quad 56 \\ 7 \quad 01 \\ \hline 2 \quad 55 \end{array} \right.
 \end{array}$$

And its time to repeat the steps again.

- a) **Bring down** the next pair of digits and **place** them next to the difference obtained in the above step.

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \\
 \hline
 \sqrt{12'34.56'00'00'00} \\
 9 \\
 \hline
 65 \quad \left| \begin{array}{r} 3 \quad 34 \\ 3 \quad 25 \end{array} \right. \\
 \hline
 701 \quad \left| \begin{array}{r} 9 \quad 56 \\ 7 \quad 01 \\ \hline 2 \quad 55 \quad 00 \end{array} \right.
 \end{array}$$

- b) **Draw a vertical line** to the left of the number: **22500**

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \\
 \hline
 \sqrt{12'34.56'00'00'00} \\
 9 \\
 \hline
 65 \quad \left| \begin{array}{r} 3 \quad 34 \\ 3 \quad 25 \end{array} \right. \\
 \hline
 701 \quad \left| \begin{array}{r} 9 \quad 56 \\ 7 \quad 01 \\ \hline 2 \quad 55 \quad 00 \end{array} \right.
 \end{array}$$

- c) Whenever you draw a vertical line next a partial dividend, multiply **whatever** number is above the radical sign by **20**. In this case that means multiplying **351** times **20** which equal **7020**. **IGNORE THE DECIMAL POINT WHEN MULTIPLYING BY 20**. This number is known as a **trial divisor**.
- d) Place the trial divisor to the left of the vertical line.

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \\
 \hline
 \sqrt{12'34.56'00'00'00} \\
 9 \\
 \hline
 65 \quad \left| \begin{array}{r} 3 \quad 34 \\ 3 \quad 25 \end{array} \right. \\
 \hline
 701 \quad \left| \begin{array}{r} 9 \quad 56 \\ 7 \quad 01 \\ \hline 2 \quad 55 \quad 00 \end{array} \right. \\
 7020 \quad \left| \begin{array}{r} 2 \quad 55 \quad 00 \end{array} \right.
 \end{array}$$

- a) **Bring down** the next pair of digits and **place** them next to the difference obtained in the above step.

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \quad 3 \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \quad | \quad 3 \quad 34 \\
 \quad \quad \underline{3 \quad 25} \\
 701 \quad | \quad 9 \quad 56 \\
 \quad \quad \quad \underline{7 \quad 01} \\
 7023 \quad | \quad 2 \quad 55 \quad 00 \\
 \quad \quad \quad \underline{2 \quad 10 \quad 69} \\
 \quad \quad \quad \quad \quad \quad \underline{44 \quad 31 \quad 00}
 \end{array}$$

- b) **Draw a vertical line** to the left of the number: **443100**

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \quad 3 \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \quad | \quad 3 \quad 34 \\
 \quad \quad \underline{3 \quad 25} \\
 701 \quad | \quad 9 \quad 56 \\
 \quad \quad \quad \underline{7 \quad 01} \\
 7023 \quad | \quad 2 \quad 55 \quad 00 \\
 \quad \quad \quad \underline{2 \quad 10 \quad 69} \\
 \quad \quad \quad \quad \quad \quad | \quad 44 \quad 31 \quad 00
 \end{array}$$

- c) Whenever you draw a vertical line next a partial dividend, multiply **whatever** number is above the radical sign by **20**. In this case that means multiplying **3513** times **20** which equal **70260**. **IGNORE THE DECIMAL POINT WHEN MULTIPLYING BY 20**. This number is known as a **trial divisor**.

- d) Place the trial divisor to the left of the vertical line.

$$\begin{array}{r}
 3 \quad 5 \cdot 1 \quad 3 \\
 \sqrt{12'34.56'00'00'00} \\
 \underline{9} \\
 65 \quad | \quad 3 \quad 34 \\
 \quad \quad \underline{3 \quad 25} \\
 701 \quad | \quad 9 \quad 56 \\
 \quad \quad \quad \underline{7 \quad 01} \\
 7023 \quad | \quad 2 \quad 55 \quad 00 \\
 \quad \quad \quad \underline{2 \quad 10 \quad 69} \\
 70260 \quad | \quad 44 \quad 31 \quad 00
 \end{array}$$

- e) Its now time to convert the trial divisor into the actual divisor. Estimate how many times the first digit in the trial divisor will go into the first two digits of the partial dividend (443100).

In this case, 7 will divide into 44, 6 times.

TO MAKE THE ACTUAL DIVISOR, YOU PLACE THE ESTIMATED NUMBER OF TIMES THE TRIAL DIVISOR WILL DIVIDE (in this case, 6) ABOVE THE RADICAL SIGN DIRECTLY CENTERED OVER THE NEXT PAIR OF DIGITS (in this case, 00) AND ADD 6 TO THE TRIAL DIVISOR. THUS, THE ACTUAL DIVISOR BECOMES 70266.

$$\begin{array}{r}
 3 5 \bullet 1 3 6 \\
 \sqrt{12'34.56'00'00'00} \\
 9 \\
 65 \overline{) 3 34} \\
 3 25 \\
 701 \overline{) 9 56} \\
 7 01 \\
 7023 \overline{) 2 55 00} \\
 2 10 69 \\
 70266 \overline{) 44 31 00}
 \end{array}$$

- f) The newly formed actual divisor **MUST** be tested. In this case, this means multiplying 6 times 70266

$$\begin{array}{r}
 \mathbf{70266} \\
 \mathbf{\times 6} \\
 \hline
 \mathbf{421596}
 \end{array}$$

In our given problem 421,596 is less than 443,100, so we are done finding this actual divisor.

- g) Once the actual divisor is determined, place the product of the actual divisor and estimated number (6 x 70266 = 421596) underneath the partial dividend and subtract.

$$\begin{array}{r}
 3 5 \bullet 1 3 6 \\
 \sqrt{12'34.56'00'00'00} \\
 9 \\
 65 \overline{) 3 34} \\
 3 25 \\
 701 \overline{) 9 56} \\
 7 01 \\
 7023 \overline{) 2 55 00} \\
 2 10 69 \\
 70266 \overline{) 44 31 00} \\
 42 15 96 \\
 \hline
 2 15 04
 \end{array}$$

And its time to repeat the steps again.

- a) **Bring down** the next pair of digits and **place** them next to the difference obtained in the above step.

$$\begin{array}{r} 35\bullet 136 \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ 65 \overline{) 334} \\ \underline{325} \\ 701 \overline{) 956} \\ \underline{701} \\ 7023 \overline{) 25500} \\ \underline{21069} \\ 70266 \overline{) 443100} \\ \underline{421596} \\ \hline 2150400 \end{array}$$

- b) **Draw a vertical line** to the right of the number: **2150400**

$$\begin{array}{r} 35\bullet 136 \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ 65 \overline{) 334} \\ \underline{325} \\ 701 \overline{) 956} \\ \underline{701} \\ 7023 \overline{) 25500} \\ \underline{21069} \\ 70266 \overline{) 443100} \\ \underline{421596} \\ \hline 2150400 \end{array}$$

- c) Whenever you draw a vertical line next a partial dividend, multiply **whatever** number is above the radical sign by **20**. In this case that means multiplying **35136** times **20** which equal **702720**. **IGNORE THE DECIMAL POINT WHEN MULTIPLYING BY 20**. This number is known as a **trial divisor**.

- d) Place the trial divisor to the left of the vertical line.

$$\begin{array}{r} 35\bullet 136 \\ \sqrt{12'34.56'00'00'00} \\ \underline{9} \\ 65 \overline{) 334} \\ \underline{325} \\ 701 \overline{) 956} \\ \underline{701} \\ 7023 \overline{) 25500} \\ \underline{21069} \\ 70266 \overline{) 443100} \\ \underline{421596} \\ 702720 \overline{) 2150400} \end{array}$$

- e) Its now time to convert the trial divisor into the actual divisor. Estimate how many times the first digit in the trial divisor will go into the first two digits of the partial dividend (2150400).

In this case, 7 will divide into 21, 3 times.

TO MAKE THE ACTUAL DIVISOR, YOU PLACE THE ESTIMATED NUMBER OF TIMES THE TRIAL DIVISOR WILL DIVIDE (in this case, 3) ABOVE THE RADICAL SIGN DIRECTLY CENTERED OVER THE NEXT PAIR OF DIGITS (in this case, 00) AND ADD 3 TO THE TRIAL DIVISOR. THUS, THE ACTUAL DIVISOR BECOMES 702723.

$$\begin{array}{r}
 \begin{array}{ccccccc}
 & 3 & 5 & \bullet & 1 & 3 & 6 & 3 \\
 \sqrt{12'34.56'00'00'00} \\
 9 & & & & & & & \\
 \hline
 65 & \overline{) 3\ 34} \\
 & 3\ 25 \\
 & \hline
 & 9\ 56 \\
 701 & \overline{) 9\ 56} \\
 & 7\ 01 \\
 & \hline
 & 2\ 55\ 00 \\
 7023 & \overline{) 2\ 55\ 00} \\
 & 2\ 10\ 69 \\
 & \hline
 & 44\ 31\ 00 \\
 70266 & \overline{) 44\ 31\ 00} \\
 & 42\ 15\ 96 \\
 & \hline
 702723 & \overline{) 2\ 15\ 04\ 00}
 \end{array}
 \end{array}$$

- f) The newly formed actual divisor **MUST** be tested. In this case, this means multiplying 3 times 702723

$$\begin{array}{r}
 702723 \\
 \times \quad 3 \\
 \hline
 2108169
 \end{array}$$

In our given problem 2,108,169 is less than 2,150,400, so we are done finding this actual divisor.

- g) Once the actual divisor is determined, place the product of the actual divisor and estimated number (3 x 702723 = 2108169) underneath the partial dividend and subtract.

$$\begin{array}{r}
 \begin{array}{ccccccc}
 & 3 & 5 & \bullet & 1 & 3 & 6 & 3 \\
 \sqrt{12'34.56'00'00'00} \\
 9 & & & & & & & \\
 \hline
 65 & \overline{) 3\ 34} \\
 & 3\ 25 \\
 & \hline
 & 9\ 56 \\
 701 & \overline{) 9\ 56} \\
 & 7\ 01 \\
 & \hline
 & 2\ 55\ 00 \\
 7023 & \overline{) 2\ 55\ 00} \\
 & 2\ 10\ 69 \\
 & \hline
 & 44\ 31\ 00 \\
 70266 & \overline{) 44\ 31\ 00} \\
 & 42\ 15\ 96 \\
 & \hline
 702723 & \overline{) 2\ 15\ 04\ 00} \\
 & 2\ 10\ 81\ 69 \\
 & \hline
 & & & 4\ 22\ 31
 \end{array}
 \end{array}$$

NOTE 1: We now have found the square root of the original number correct to four places.

NOTE 2: The final difference (42231) is important only if you wish to CHECK your answer.